

# Chern-Simons theory on $S^1$ -bundles: abelianisation and q-deformed Yang-Mills theory

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**ABSTRACT:** We study Chern-Simons theory on 3-manifolds  $M$  that are circle-bundles over 2-dimensional surfaces  $\Sigma$  and show that the method of Abelianisation, previously employed for trivial bundles  $\Sigma \times S^1$ , can be adapted to this case. This reduces the non-Abelian theory on  $M$  to a 2-dimensional Abelian theory on  $\Sigma$  which we identify with q-deformed Yang-Mills theory, as anticipated by Vafa et al. We compare and contrast our results with those obtained by Beasley and Witten using the method of non-Abelian localisation, and determine the surgery and framing prescription implicit in this path integral evaluation. We also comment on the extension of these methods to BF theory and other generalisations.

**KEYWORDS:** Chern-Simons Theories, Topological Field Theories.

*Dedicated to Bob Delbourgo on the occasion of his 65th Birthday.*

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## 1. Introduction

There has been a great success in computing the Chern-Simons partition function by using conformal field theory techniques combined with surgery. On the other hand there are very few exact (i.e. non-perturbative) gauge theory path integral computations of Chern-Simons theory, the exceptions being on manifolds of the form  $\Sigma_g \times S^1$  [1]. The technique adopted in [1], and reviewed in [2, 3], was Abelianisation of the non-Abelian theory.

In this paper, we will study Chern-Simons theory on 3-manifolds  $M_{(g,p)}$  that are non-trivial (monopole degree  $p$ ) circle bundles over 2-dimensional (genus  $g$ ) surfaces. Thus  $M_{(g,0)} = \Sigma_g \times S^1$ ,  $M_{(0,p)} = L(p, 1)$  are Lens spaces etc.

There are three principal reasons for us for looking at this issue:

- From a path integral technological point of view, the question arises, if the method of Abelianisation can be generalised from trivial ( $p = 0$ ) to non-trivial circle bundles: Abelianisation works well in two dimensions; thus one needs to be able to “push down” things from the 3-manifold to the base, which is a somewhat less obvious procedure in the case of non-trivial bundles.

- Chern-Simons theory on the 3-manifolds  $M_{(g,p)}$  (and more general Seifert manifolds) has recently been studied by Beasley and Witten [4] using the method of non-Abelian localisation [5]. In this context the question arises whether the diagonalisation procedure (once one has established that it is applicable) yields results that are manifestly equivalent or comparable to those of [4] and if there are situations in which one or the other method is more efficient.
- Chern-Simons theory on Lens spaces  $L(p, 1) = M_{(0,p)}$  has also recently appeared in the context of black hole partition function calculations via topological string theory [6, 7]. There, methods of Abelianisation were used to argue that this theory is equivalent to a “q-deformed” two-dimensional Yang-Mills theory. In [7] the connection with Chern-Simons theory was somewhat indirect and the question arises if it is possible to derive the relation between the action of Chern-Simons theory and that of a two-dimensional action in a direct manner.

The Chern-Simons action is

$$kS_{CS}[A] = \frac{k}{4\pi} \int_M \text{Tr} \left( AdA + \frac{2}{3}A^3 \right) . \tag{1.1}$$

In order to get a handle on the cubic part of the Chern-Simons action, we make use of the geometry of  $M_{(g,p)}$ , following [4], to decompose the connection into a horizontal and a vertical part, which now appear at most quadratically in the action and thus lend themselves to a path integral treatment.<sup>1</sup>

In particular, due to the non-triviality of the bundle one finds a term quadratic in the vertical component of the connection (a scalar  $\phi$  from the point of view of the base) from the quadratic term of the Chern-Simons action, suggesting, already at this stage, a relation with some kind of 2-dimensional Yang-Mills theory rather than with a BF- or  $G/G$ -like theory (as encountered for  $p = 0$  in [1]). This new term apart, the action resembles that of Chern-Simons theory on a trivial bundle  $\Sigma_g \times S^1$ , and we can now attempt to apply the methods of [1] to this case.

We can summarise the results that we find as follows:

**Diagonalisation.** The method of diagonalisation is applicable to the case of non-trivial circle bundles and permits one to reduce the non-Abelian 3-dimensional Chern-Simons theory to a 2-dimensional Abelian theory (whose partition function can in many cases be evaluated in a straightforward manner).

The main differences to the previously discussed case of trivial  $S^1$ - bundles  $M_{(g,0)}$  are:

- the obstruction bundles to diagonalisation (which the method of Abelianisation instructs one to sum over in the path integral) are now precisely the torsion bundles on the 3-manifold  $M_{(g,p)}$ .

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<sup>1</sup>In contrast to [4], however, we do not introduce a corresponding Stückelberg field and shift symmetry associated with this decomposition.

- the fields that one integrates out, in the process of reducing the theory from 3 to 2 dimensions (the non- $U(1)$ -invariant fields), are now sections of non-trivial bundles  $\mathcal{O}(-np)$  on the base  $\Sigma_g$ ; correspondingly this changes the evaluation of the determinants.

The abelianised expression for the partition function  $Z_k$  of the level  $k$  Chern-Simons theory with gauge group  $G$  we find is<sup>2</sup>

$$Z_k[M_{(g,p)}, G] = \sum_{r \in \mathbb{Z}^{\mathbf{rk}}_p} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (1.2)$$

The integral over the Cartan subalgebra  $\mathfrak{t}$  of  $G$  is all that remains of the integral over the vertical component of the gauge field, the integral over all other modes and fields, in particular the horizontal components of the gauge fields, having already been performed. Here  $\mathbf{rk} = \dim \mathfrak{t}$  is the rank of  $G$  and  $\chi(\Sigma_g) = 2 - 2g$  the Euler characteristic of  $\Sigma_g$ . The Ray-Singer torsion  $T_{S^1}$  of  $S^1$  and the shift  $k \rightarrow k + c_{\mathfrak{g}}$  arise from the absolute value and phase of the ratio of determinants generated by the integral over these gauge field modes and the ghosts. The sum over  $r$  is the sum over the torsion classes of line bundles mentioned above.

**Comparison with non-abelian localisation.** In a recent paper, Beasley and Witten [4] adapted the method of non-Abelian localisation (originally developed for 2-dimensional Yang-Mills theory in [5]) to Chern-Simons theory on Seifert manifolds.

The upshot of this localisation is that the partition function can schematically be written as sums of integrals of the form

$$Z_k[M, G] = \sum \int_{\mathfrak{g} \times \mathcal{M}} \mathcal{F} = \sum \int_{\mathfrak{t} \times \mathcal{M}} \widehat{\mathcal{F}} \quad (1.3)$$

for some integrand  $\mathcal{F}$ . Here  $\mathfrak{g}$  is the Lie algebra of  $G$  and  $\mathcal{M}$  is the component of the moduli space (of flat or Yang-Mills connections, say) onto which the theory localised, and the right hand side is what one gets on applying the Weyl integral formula to reduce the Lie algebra integral to the Cartan subalgebra  $\mathfrak{t}$ .

One of the motivations for [4] was to explain formulae of Rozansky and Lawrence [8] in the  $SU(2)$  case. Following previous work of Rozansky, Lawrence and Rozansky begin with the conformal field theory formula for the Chern-Simons partition function in terms of characters of integrable representations of  $SU(2)$ . The empirical discovery here was that the Chern-Simons partition function could be expressed as coming from stationary phase contributions to the path integral, that is as integrals and residues over the moduli space of flat connections on  $M$ . The localisation of [4] explains in an a priori manner why this is so.

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<sup>2</sup>For  $G$  simply-laced and simply-connected; in this introductory section we will also suppress certain overall normalisation and phase factors from the equations.

A prototypical example are the Lens spaces  $L(p, 1)$ . For  $G = SU(2)$ , the formula of [8] is, in the normalisation of [4],

$$Z_k[M_{(0,p)}, SU(2)] \sim \sum_{r=0}^{p-1} \frac{1}{2\pi i} \int_{C^{(r)}} dz \left(2 \sinh \frac{z}{2}\right)^2 \exp\left(\frac{i(k+2)}{8\pi} pz^2 - (k+2)rz\right) \quad (1.4)$$

(we have suppressed an overall framing dependent phase) where the contour  $C^{(0)} = e^{\frac{i\pi}{4}} \times \mathbb{R}$  and the other contours  $C^{(r)}$  are parallel to this one through the stationary phase point  $z = -4\pi ir/p$ . This is of the general form (1.3), the sum being a sum over the flat connections on  $L(p, 1)$ . For the generalisation of such formulae to  $SU(n)$ ,  $n > 2$ , see [9].

The formulae that we derive, on the other hand, take the general form (cf. (1.2))

$$Z_k[M_{(g,p)}, G] = \sum_{\mathfrak{t}} \int_{\mathfrak{t}} \tilde{\mathcal{F}}. \quad (1.5)$$

This agrees with (1.3) when  $\mathcal{M}$  is a point or a finite union of points. In particular, (1.4) is also of the general form (1.5) and it is easy to see (using the explicit form of the Ray-Singer torsion and analytic continuation - there are no poles to worry about) that our general result (1.2) reproduces (1.4).

More generally, however, a striking difference between (1.3) and (1.5) is the absence of an integration over  $\mathcal{M}$  from the latter. This is a general feature of our strategy for solving low-dimensional gauge theories, e.g. BF theory, via path integrals. In the latter case, by integrating over  $B$  first, one sees that one is left with an integral over the moduli space of flat connections and is essentially calculating its volume. This integral is typically difficult to perform in practice. By reversing the order of integration, however, i.e. integrating first over all connections and only then over  $B$ , one completely side-steps the issue of having to integrate over the moduli space of flat connections and obtains explicit expressions for the volume - see e.g. [2]. Adopting the same strategy here, we arrive at (1.2), which is of the general form (1.5), with the integral over  $\mathcal{M}$  having, somewhat miraculously, been taken care of. Since the moduli spaces in question can be quite nasty and singular, not having to work with them directly is a blessing.

**Relation with qYM-theory.** An intermediate step in arriving at the final formula (1.2) is the reduction of Chern-Simons theory on  $M_{(g,p)}$  to an effective 2-dimensional Abelian theory. This theory, given by (4.5), with action (4.7),

$$S_{\Sigma}[A_H, \phi] = \frac{k + c_{\mathfrak{g}}}{4\pi} \int_{\Sigma_g} \text{Tr}(2\phi F_H + p\phi^2\omega), \quad (1.6)$$

and the gauge field integration range as specified in (5.6), is of the abelianised BF- or  $G/G$ -model type with an additional  $\phi^2$ -interaction. This theory can be considered as a deformation of ordinary Yang-Mills theory, the deformation residing in the measure and the finite sum over torsion bundles. Since the theory that one obtains for  $p = 0$ , namely the  $G/G$ -model and its Abelianisation, can be interpreted as a q-deformed BF theory (in

the sense that in the expression for the partition function dimensions of representations are replaced by their quantum-dimensions - see e.g. [1]), it is natural to suspect that what one will find for  $p \neq 0$  is a corresponding q-deformation of Yang-Mills theory.

In [7], on the other hand, an alternative deformed 2-dimensional Yang-Mills description of Chern-Simons theory on Lens spaces  $M_{(0,p)}$  was proposed, involving an action of the above form, but with a compact scalar (so that the  $\phi^2$ -term in the action requires some interpretation) and an infinite sum over torus bundles, both in apparent contrast with (1.2), (1.4). The partition function of this theory for general  $M_{(g,p)}$  is

$$\tilde{Z}_k[M_{(g,p)}, G] = \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}/I} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (1.7)$$

where  $I$  is the integral lattice (so that  $\phi$  is now compact) and the sum over line bundles is not constrained. It was shown in [7] that this partition function bears the same relation to q-deformed representation theory as ordinary Yang-Mills theory does to ordinary representation theory, and thus the theory can legitimately be referred to as q-deformed Yang-Mills theory.

We will show in section 6 that, despite appearance, (1.7) agrees precisely with our result (1.2),

$$\tilde{Z}_k[M_{(g,p)}, G] = Z_k[M_{(g,p)}, G], \quad (1.8)$$

so that the q-deformation can be equivalently regarded as arising from either a compact scalar or a restricted sum over torus bundles, the latter description arising more naturally from the point of view of diagonalisation.

In particular, this shows that the partition function on  $M_{(g,p)}$  is the same as the expectation value of an operator in the  $G/G$ -model, i.e. in the theory on  $M_{(g,0)} = \Sigma_g \times S^1$  where  $\phi$  is naturally compact due to large gauge transformations. Concretely, up to a phase proportional to  $p$  (which we calculate), one has

$$Z_k[M_{(g,p)}, G] = \left\langle e^{ip \frac{(k+c_{\mathfrak{g}})}{4\pi} \text{Tr} \phi^2} \right\rangle_{\Sigma_g \times S^1}. \quad (1.9)$$

This means that manifolds of non-trivial Chern classes are simply created by insertions of the operator  $\exp i \frac{(k+c_{\mathfrak{g}})}{4\pi} \text{Tr} \phi^2$  in the path integral for the trivial bundle. This has been argued before by Vafa [6] in a rather different way. Our calculation provides an *ab initio* path integral derivation of this cute fact.

For earlier work on q-deformed Yang-Mills theory see [10] and, in particular, [11] where a Lagrangian realisation of this theory was proposed and solved by the method of diagonalisation. Aspects of the relation between Chern-Simons theory on Lens spaces and Yang-Mills theory were previously discussed in [12]. For other recent work on q-deformed Yang-Mills theory see [13, 14].

**Comparison with the surgery prescription.** The explicit formulae that we obtain for the partition function in section 7 have the form that one would find on performing surgery

on knots and links in  $\Sigma_g \times S^1$ . Indeed, keeping track of all phases, the partition function can be written in terms of the standard modular  $S$ - and  $T$ -matrices of the Wess-Zumino-Witten model as

$$Z_k[M_{(g,p)}, G] = \sum_{\lambda} S_{0\lambda}^{2-2g} T_{\lambda}^{-p}, \tag{1.10}$$

where the sum is over level  $k$  integrable weights.

Quite generally, starting from  $M_{(g,0)}$  different surgeries can yield the same 3-manifold  $M_{(g,p)}$  but this manifold will come equipped with a framing which depends on the surgery so the formulae we obtain involve an implicit choice of framings. The evaluation of the path integral then is always in some framing of the 3-manifold (and links) in question. However, it is not at all transparent from the outset which framing one is actually in.

From their results, Beasley and Witten [4] deduce that, in their calculations, they are in the ‘Seifert’ framing and not in the canonical framing of the 3-manifold. Starting from  $S^2 \times S^1$  one generates  $S^3$ , by acting with  $T^m ST^n$ . The canonical framing for  $S^3$  corresponds to  $m = n = 0$ , while one obtains a  $U(1)$ -invariant ‘Seifert’ framing for  $n + m = 2$ .

Our previous calculations [1] were in the canonical framing for  $M_{(g,0)}$ , so one can ask which surgery prescription is being used to generate the  $M_{(g,p)}$  with the framing that is employed in our path integral. By considering only the partition function we guess that the surgery prescription we are implicitly using to get from  $M_{(g,0)}$  to  $M_{(g,p)}$  is to act with  $(TST)^p$ .

The surgery prescription is always such that the expectation value of the Hopf link equals the surgery matrix. In [1] Abelianisation was employed to also compute expectation values of Wilson lines in the  $S^1$  direction of  $M_{(g,0)}$ . In section 8 we extend this to compute the expectation values of Wilson lines in the non-trivial fibre direction of  $M_{(g,p)}$ . In particular, we show that the expectation value of the Hopf link is indeed  $TST$ , confirming the guess that we made.

**Generalisations.** We have not aimed for maximal generality in this article, and the results can be generalised in various ways, to other three-manifolds, other groups, and other 3-dimensional gauge theories (such as BF theory). We will briefly come back to these issues in section 9.

## 2. Gauge theories on 3-manifolds $M_{(g,p)}$

We will consider gauge theories (and later on more specifically Chern-Simons theory) on 3-manifolds  $M_{(g,p)}$  which are themselves principal  $U(1)$  bundles  $U(1) \rightarrow M_{(g,p)} \xrightarrow{\pi} \Sigma_g$  over 2-dimensional surfaces  $\Sigma_g$  of genus  $g$  and first Chern (or Euler) class  $-p \in \mathbb{Z}$  (under the identification  $H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$ ; the minus sign is a consequence of our conventions which we spell out below).

We choose the gauge group  $G$  to be compact, (semi-)simple, and simply connected. In particular, this implies that a principal  $G$ -bundle on a 3-manifold  $M$  and all its associated vector bundles are trivial. In this case we understand the global obstructions to diagonalisation well enough to be able to apply this method to gauge theories on  $M$ . For comments

on the more general case we refer to [15] and the discussion in section 9. We also assume, for convenience and notational simplicity, that  $G$  is simply laced even though this latter assumption is not necessary.

The analysis of gauge theories on such 3-manifolds  $M_{(g,p)}$  is greatly simplified by appropriate gauge choices that are adapted to the geometry of the situation at hand. In the case  $p = 0$ , i.e.  $M = M_{(g,0)} = \Sigma_g \times S^1$  [1], it was natural to single out the “vertical” component of the connection,  $A_\theta$  say, with  $\theta$  an angular fibre coordinate, and to impose, as a first step, the condition

$$\partial_\theta A_\theta = 0 \tag{2.1}$$

(the simpler axial gauge condition  $A_\theta = 0$  not being available because of the possibility of having non-trivial holonomy along the  $S^1$ ). In a second step, it was then possible (and very effective) to use the residual 2-dimensional gauge invariance to “diagonalise”  $A_\theta$ , i.e. to conjugate it into the Cartan subalgebra  $\mathfrak{t}$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . This is tantamount to imposing the condition

$$A_\theta^\mathfrak{k} = 0 \tag{2.2}$$

where  $\mathfrak{k}$  is the orthogonal complement of  $\mathfrak{t}$  in  $\mathfrak{g}$ ,  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ , with respect to the Killing-Cartan form.

In order to mimic this gauge fixing procedure for  $p \neq 0$ , we want to again single out one particular component of the gauge field. This amounts to picking a one dimensional sub-bundle of the cotangent bundle  $T^*M$  of  $M$ . For  $M = M_{(g,p)}$  there is a natural way to do this.

**Geometric set-up.** Indeed, let  $\kappa$  be a connection on the principal  $U(1)$ -bundle  $M_{(g,p)}$ , thought of as a globally defined real-valued 1-form on the total space of the bundle, and denote by  $K$  the fundamental vector field on  $M_{(g,p)}$ , i.e. the generator of the  $U(1)$ -action. A connection  $\kappa$  is characterised by

$$\iota_K \kappa = 1 \tag{2.3}$$

and the equivariance condition

$$L_K \kappa = 0, \tag{2.4}$$

where  $L_K = \{d, \iota_K\}$  is the Lie derivative in the  $K$  direction. These two conditions imply that  $\iota_K d\kappa = 0$ , i.e. the expected statement that the curvature 2-form  $d\kappa$  of  $\kappa$  is horizontal.

Note that  $\kappa$  is not unique. In local coordinates one has

$$\kappa = d\theta + a, \tag{2.5}$$

where  $\theta$  is a fibre coordinate,  $0 \leq \theta < 1$ , and  $a = a_i dx^i$  is a local representative on  $\Sigma_g$  of the connection  $\kappa$  on  $M_{(g,p)}$ . Since  $M_{(g,p)}$  has degree  $p$ , we may choose  $\kappa$  (and hence  $a$ ) so that the curvature 2-form satisfies

$$d\kappa = p \pi^*(\omega) \tag{2.6}$$

for  $\omega$  a unit normalised symplectic form on  $\Sigma_g$ ,

$$\int_{\Sigma_g} \omega = 1. \tag{2.7}$$

In passing we note that for  $p \neq 0$  a choice of  $\kappa$  equips  $M_{(g,p)}$  with what is known as a contact structure, i.e. a 1-dimensional sub-bundle of  $T^*M$  (generated by a 1-form  $\kappa$ ) such that  $\kappa \wedge d\kappa$  is nowhere vanishing on  $M$ . Indeed, with the above choices we see that

$$\kappa \wedge d\kappa = p d\theta \wedge \pi^*(\omega) \tag{2.8}$$

is nowhere vanishing as required providing that the  $U(1)$  bundle is non-trivial, that is providing  $p \neq 0$ . For later use we note that

$$\int_M \kappa \wedge d\kappa = p \int_{\Sigma_g} \omega = p. \tag{2.9}$$

Thus, depending on the sign of  $p$ ,  $\int_M \kappa \wedge d\kappa$  may be either positive or negative. In this respect our conventions differ from those of [4] where the orientation of  $M$  is chosen such that  $p$  is non-negative.

Contact structures can be put on any compact orientable 3-manifold [16] but, as we have seen, are particularly simple to describe when  $M$  is a principal  $U(1)$ -bundle. This contact structure point of view played an important role in the considerations of [4]. In the present paper, however, we will downplay the role of the contact structure somewhat in order to bring out the analogy with the case  $p = 0$  for which the above construction fails to provide a contact structure.

Our convention for  $U(1)$  is that the generator of its Lie algebra is  $i$ . A connection on a  $U(1)$  bundle is locally

$$\varpi = 2\pi i\kappa = 2\pi i(a + d\theta). \tag{2.10}$$

Chern classes are generated by  $\det(I - \frac{1}{2\pi i}d\varpi)$  so that  $c_1 = [-d\kappa]$ . Consequently the first Chern class of the naturally associated line bundle to  $M_{(g,p)}$  is

$$- \int_{\Sigma_g} d\kappa = -p. \tag{2.11}$$

Comparing with the discussion in [17, p.121], we see that  $\kappa$  is precisely what is called an angular form there (and denoted by  $\psi$ ): this form has the property that the vertical component is the unit volume form, with the standard orientation, and that its exterior derivative is minus the pull back of the Euler class, or first Chern class, of the associated bundle;  $\kappa$  has these properties since its vertical component is  $d\theta$  and  $d\kappa = -\pi^*(-p\omega)$ .

**Decomposition.** As both  $K$  and  $\kappa$  are nowhere vanishing we have, by virtue of (2.3), that  $\kappa \wedge \iota_K$  and  $(1 - \kappa \wedge \iota_K)$  are projection operators, corresponding to the decomposition

$$T^*M = T^*_\kappa(M) \oplus T^*_H(M), \quad T^*_\kappa(M) \approx \Omega^0(M) \tag{2.12}$$

into forms along the  $\kappa$  direction and those which are horizontal. Concretely, for  $\alpha \in \Omega^1(M, \mathbb{R})$  one has  $\alpha = \alpha_\kappa + \alpha_H$  with

$$\alpha_\kappa = \kappa \wedge \iota_K \alpha \in \Omega_\kappa^1(M, \mathbb{R}), \quad \alpha_H = (1 - \kappa \wedge \iota_K) \alpha \in \Omega_H^1(M, \mathbb{R}). \quad (2.13)$$

Likewise we can decompose connections on vector bundles  $E$  over  $M$ , thought of as elements of  $\Omega^1(M, \mathfrak{g})$ ,

$$A = A_\kappa + A_H \equiv \phi \kappa + A_H. \quad (2.14)$$

Since  $\phi \in \Omega^0(M, \mathfrak{g})$  we can think of it as a section of the adjoint bundle  $E = M \times \mathfrak{g}$ . Its transformation behaviour under infinitesimal gauge transformations  $\delta A = d_A \Lambda$  is

$$\delta \phi = \mathcal{L}_\phi \Lambda, \quad (2.15)$$

with

$$\mathcal{L}_\phi = L_K + [\phi, \cdot]. \quad (2.16)$$

There is also a decomposition of the exterior derivative

$$d = (1 - \kappa \wedge \iota_K) d + \kappa \wedge \iota_K d \equiv \pi^* d_\Sigma + d_K. \quad (2.17)$$

On horizontal forms  $B_H$ ,  $\iota_K B_H = 0$ , one has  $\iota_K dB_H = L_K B_H$ , with  $L_K B_H$  also horizontal, and therefore

$$dB_H = (\pi^* d_\Sigma) B_H + \kappa \wedge L_K B_H, \quad (2.18)$$

and, in particular,

$$B_H \wedge dB_H = B_H \wedge \kappa \wedge L_K B_H. \quad (2.19)$$

From (2.4) we also have the useful fact

$$\kappa \wedge L_K B_H = L_K(\kappa \wedge B_H). \quad (2.20)$$

**Gauge choices.** Having singled out a particular component of the gauge field  $A_\kappa$  it is tempting to impose the gauge condition  $A_\kappa = 0 = \phi$ . However, just as for  $p = 0$ , this is not possible since Wilson loops along the fibres of  $M_{(g,p)} \rightarrow \Sigma_g$  are gauge invariant and non-trivial (we will discuss their correlation functions in Chern-Simons theory in section 8).

Instead we may (and do) impose the analogue of the condition (2.1), namely

$$L_K A_\kappa = 0 \Leftrightarrow L_K \phi = \iota_K d \phi = 0. \quad (2.21)$$

This gauge condition,  $L_K \phi = 0$ , tells us that  $\phi$  is a  $U(1)$ -invariant section of  $E$ . Equivalently, it can therefore be regarded as a section of the (trivial) adjoint bundle  $V$  over  $\Sigma_g$  (see appendix A).

Having pushed down  $\phi$  to  $\Sigma_g$  in this manner, we can now proceed to the diagonalisation of  $\phi$  as in [1]. Thus let  $T$  be some maximal torus of  $G$  and  $\mathfrak{t}$  the corresponding Cartan subalgebra, with  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ . We now impose the analogue of (2.2), namely

$$\phi^{\mathfrak{k}} = 0. \tag{2.22}$$

As shown in [1, 15], the price for diagonalising sections of  $V$  is that in the path integral, when we come to it, we must sum over all  $T$ -bundles on  $\Sigma_g$ , hence from the 3-dimensional perspective over all  $T$ -bundles that one gets by pull back from  $\Sigma_g$ . Since the pull-back  $\pi^*M_{(g,p)}$  of the  $U(1)$ -bundle  $M_{(g,p)} \xrightarrow{\pi} \Sigma_g$  to the total space  $M_{(g,p)}$  is (tautologically) trivial,  $\pi^*M_{(g,p)} = M_{(g,p)} \times U(1)$ , the pull-back of the  $p$ -th power of any line bundle on  $\Sigma_g$  to  $M_{(g,p)}$  is trivial. Thus the pull-backs of line bundles from  $\Sigma_g$  to  $M_{(g,p)}$  are of finite order. We show in appendix A that all torsion (finite order  $p$ ) bundles on  $M_{(g,p)}$  arise in this way, so that it is precisely these bundles that we should sum over in the path integral.

**Ghost action.** The BRST symmetry of the gauge theory is standard and we do not repeat it here. We mimic arguments presented in detail in [1] for fixing the gauge and the associated ghost terms. Both of the conditions (2.21) and (2.22) can be simultaneously imposed by adding the BRST exact terms

$$\int_M [E * \phi + \bar{c} * \mathcal{L}_\phi c] = \int_M [E * \phi + d\kappa \wedge \kappa \bar{c} \mathcal{L}_\phi c] \tag{2.23}$$

with the understanding that those modes which are  $U(1)$ -invariant, i.e. solutions to the equations

$$L_K E^{\mathfrak{t}} = L_K c^{\mathfrak{t}} = L_K \bar{c}^{\mathfrak{t}} = 0, \tag{2.24}$$

are not to be included in the path integral.

Here  $*$  refers to a metric  $g_M$  on  $M$ . It is convenient to choose this metric to be  $U(1)$ -invariant, and a natural choice (which we will adopt) is

$$g_M = \pi^* g_\Sigma + \kappa \otimes \kappa \tag{2.25}$$

with  $g_\Sigma$  a metric on  $\Sigma_g$  such that  $*_{\Sigma} 1 = \omega$  (2.7).

Thus far the discussion has not been theory specific. All of the considerations above could be applied to, say, Yang-Mills theory on  $M$ . In the next section we move on to the theory of interest for us.

### 3. Chern-Simons theory on $M_{(g,p)}$

The level  $k$  Chern-Simons action is

$$kS_{CS}[A] = \frac{k}{4\pi} \int_M \text{Tr} \left( AdA + \frac{2}{3} A^3 \right) \tag{3.1}$$

In terms of the decomposition (2.14) the integrand becomes

$$\text{Tr} (A_H \wedge d_\phi A_H + \phi \kappa \wedge dA_H + \phi d\kappa \wedge A_H + \kappa \wedge A_H \wedge d\phi + \phi^2 \kappa \wedge d\kappa) \tag{3.2}$$

where

$$d_\phi = d + \kappa \wedge [\phi, \ ] . \tag{3.3}$$

Note in particular the appearance of a term quadratic in  $\phi$  for  $p \neq 0$ .

The  $G$ -bundles we are considering are trivial so we may take  $A$  to be a Lie algebra valued form. This means that, up to a total derivative, we can rewrite (3.2) as

$$\text{Tr} (A_H \wedge d_\phi A_H + 2\phi \kappa \wedge d A_H + \phi^2 \kappa \wedge d \kappa) . \tag{3.4}$$

Since the forms  $A_H$  are orthogonal to  $\kappa$ , the first term necessarily only involves a derivative in the direction of  $\kappa$ ,

$$\text{Tr} A_H \wedge d_\phi A_H = \text{Tr} A_H \wedge \kappa \wedge \mathcal{L}_\phi A_H \tag{3.5}$$

(while the derivative in the second term acts only in the horizontal direction) and thus we can write the action as

$$k S_{CS}[A_H, \phi] = \frac{k}{4\pi} \int_M \text{Tr} (A_H \wedge \kappa \wedge \mathcal{L}_\phi A_H + 2\phi \kappa \wedge d A_H + \phi^2 \kappa \wedge d \kappa) . \tag{3.6}$$

**Conditions on  $\phi$ .** Consider those  $A_H^t$  which are  $U(1)$  invariant,

$$L_K A_H^t = 0 . \tag{3.7}$$

It follows from (3.5) that these fields do not appear in the kinetic term  $A_H \wedge d_\phi A_H$ . Consequently they only appear in the mixed kinetic term  $2\phi \kappa \wedge d A_H$ . The path integral over such  $A_H^t$  then imposes a (delta function) condition on  $\phi$ , namely

$$\iota_K d(\kappa \phi) = 0 . \tag{3.8}$$

This delta function constraint on  $\phi$  together with the gauge condition (2.21) imply that  $\phi$  is actually constant,

$$d\phi = 0 . \tag{3.9}$$

We will come back to this argument and its consequences in section 5. Finally, with  $\phi$  constant we have, from (2.9), that

$$\int_M \text{Tr} \kappa \wedge d\kappa \phi^2 = p \text{Tr} \phi^2 \tag{3.10}$$

#### 4. Reduction to an abelian theory on $\Sigma_g$

Having discussed the effect of integrating out the  $U(1)$ -invariant modes of  $A_H^t$ , we now keep these and investigate what happens upon integrating out the other modes and fields, with the understanding that  $\phi$  will ultimately turn out to be constant. All these fields appear quadratically in the action, and therefore will give rise to ratios of determinants (whose definition and regularisation we will subsequently discuss in detail in appendix B).

Given the choice of metric (2.25), the operator  $*\kappa \wedge \mathcal{L}_\phi$  acts on the space of horizontal  $\mathfrak{k}$ -valued 1-forms,

$$*\kappa \wedge \mathcal{L}_\phi : \Omega_H^1(M, \mathfrak{k}) \rightarrow \Omega_H^1(M, \mathfrak{k}). \quad (4.1)$$

Hence integrating over the  $\mathfrak{k}$ -components of the ghosts  $(c^\mathfrak{k}, \bar{c}^\mathfrak{k})$  and the connection  $A_H^\mathfrak{k}$ , one obtains the following ratio of determinants:

$$\frac{\text{Det}(i\mathcal{L}_\phi)_{\Omega^0(M, \mathfrak{k})}}{\sqrt{\text{Det}(*\kappa \wedge i\mathcal{L}_\phi)_{\Omega_H^1(M, \mathfrak{k})}}}. \quad (4.2)$$

Note that

$$*\kappa = -\iota_K * = \star_2 \quad (4.3)$$

where  $\star_2$ , introduced in [4], is a lift of the Hodge duality operator  $*_\Sigma$  to  $M_{(g,p)}$  in the sense that  $\star_2 \pi^* = \pi^* *_\Sigma$ . This Hodge operator therefore appears naturally in our evaluation of the path integral and the definition of the determinants (appendix B).

Integration over the ghosts  $(c^\mathfrak{k}, \bar{c}^\mathfrak{k})$  and those  $A_H^\mathfrak{k}$  modes which are not  $U(1)$  invariant give the following ratio of determinants:

$$\frac{\text{Det}'(iL_K)_{\Omega^0(M, \mathfrak{t})}}{\sqrt{\text{Det}'(*\kappa \wedge iL_K)_{\Omega_H^1(M, \mathfrak{t})}}}. \quad (4.4)$$

The notation  $\text{Det}'$  indicates that the zero mode of the operator is not included.

On integrating out all the  $\mathfrak{k}$ -valued fields as well as all the  $\mathfrak{t}$ -valued modes which are not  $U(1)$  invariant, the Chern-Simons path integral essentially reduces to the path integral of an Abelian 2-dimensional gauge theory on  $\Sigma_g$ . Assembling all the ingredients, this path integral is

$$Z_k[M_{(g,p)}, G] \sim e^{4\pi i p \Phi_0} \int D\phi D A_H^\mathfrak{t} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp\left(i \frac{k + c_\mathfrak{g}}{4\pi} S_M\right). \quad (4.5)$$

The action is

$$S_M = \int_M \text{Tr}(2\phi \kappa \wedge F_H + \phi^2 \kappa \wedge d\kappa), \quad (4.6)$$

and since the path integral is only over invariant modes, we can push the action  $S_M$  down to  $\Sigma_g$ . Explicitly, the 2-dimensional Abelian action reads, recalling (2.6),

$$S_M \rightarrow S_\Sigma[A_H, \phi] = \frac{k + c_\mathfrak{g}}{4\pi} \int_{\Sigma_g} \text{Tr}(2\phi F_H + p\phi^2 \omega), \quad (4.7)$$

where  $A_H = A_H^\mathfrak{t}$  and  $\phi = \phi^\mathfrak{t}$ .

The various new terms appearing in (4.5) arise as follows:

The ratios of determinants (4.2, 4.4) that appear are almost unity. In calculating these ratios of determinants we pay attention to the absolute value and to the phase. As far as the absolute value is concerned, the deviation from unity is due to the mismatch in

zero modes. This mismatch is just the Euler characteristic  $\chi(\Sigma_g)$ . The Euler characteristic appears from an index theorem when we regularise as in [1] with the  $\zeta$ -function associated to the Dolbeault operator, which is rather natural given the appearance of the complex structure  $\star_2$ .

In this way, for the absolute value of the determinants one finds  $T_{S^1}(\phi)^{\chi(\Sigma_g)/2}$  where

$$T_{S^1}(\phi) = \det_{\mathfrak{k}}(1 - \text{Ad } e^{\phi}) \tag{4.8}$$

is the Ray-Singer torsion of  $S^1$  (with respect to the flat connection  $2\pi i\phi d\theta$ ).

On the other hand, when it comes to the phase, one does an  $\eta$ -function calculation. This calculation gives us the famous shift in the level,  $k \rightarrow k + c_{\mathfrak{g}}$  as well as the (framing dependent) phase  $4\pi p\Phi_0$  where

$$\Phi_0 = \frac{1}{48} \dim G. \tag{4.9}$$

There are some other things about (4.5) that require comment.

The first is that, in writing (4.5) and (4.8) we have not kept track of the overall real normalisation of the path integral (while we have kept track of the phase). For instance, in the path integral we should also integrate over harmonic  $A_H^{\mathfrak{k}}$ -modes even though these do not appear in the action. Fortunately, such modes are compact thanks to the residual Abelian gauge symmetry, as explained in [1], and consequently those modes give a finite volume factor to the path integral. We will fix the remaining real normalisation constant in section 7 by comparison with the known normalisation for  $p = 0$  [1].

The second comment is that (4.5) as it stands is incomplete as we have not specified the bundles whose connections one is to integrate over. Just as for the path integral on  $\Sigma_g \times S^1$ , upon diagonalisation, one must sum over those non-trivial  $U(1)$ -bundles that arise as obstructions to diagonalisation. As we have argued, these are precisely the torsion bundles on  $M_{(g,p)}$ . We deal with the question of how to implement this concretely and other related issues in the next section.

Before turning to these questions we wish to compare what we have done here with the calculations for  $p = 0$  in [1]. In reducing Chern-Simons theory on  $\Sigma_g \times S^1$  to an Abelian theory on  $\Sigma_g$ , one can either diagonalise first and then reduce to 2 dimensions (this is also the strategy that we have adopted here) or one can reduce first to a non-Abelian 2-dimensional theory (the  $G/G$  model) and then apply Abelianisation. In [1] we performed the detailed calculations of the determinants for the latter approach. One finds that, when the determinants arising from the (chiral, 2-dimensional)  $G/G$ -model are  $\zeta$ -function regularised, they give not only the Ray-Singer torsion (arising in this approach from the Weyl integral formula) but also the phase shift in the level,  $k \rightarrow k + c_{\mathfrak{g}}$ , together with some normalisation terms. Had we done the calculation the other way around, starting from the Abelian (and less chiral) 3-dimensional theory, then we would have found that the shift in the level arises not from the  $\zeta$ -function but from the  $\eta$ -function regularisation of the phase of the determinant accompanying the Ray-Singer torsion. Indeed the calculations of appendix B are valid for  $p = 0$  as well and thus complete this alternative calculation, only sketched in [1]. A similar calculation has also been carried out in [4].

It should be of interest to find an analogue of the second procedure (reduce first and then diagonalise) also for  $p \neq 0$ , as this would give a non-abelianised description of  $q$ -deformed Yang-Mills theory, i.e. a  $\phi F \rightarrow \phi F + \phi^2$ -like deformation of the  $G/G$ -model, perhaps the Lagrangian realisation proposed in [11].

### 5. The resulting abelian theory

The Abelian curvature 2-form  $F_H$  in (4.7) involves not only  $A_H^t$  but also a component of the connection in the  $\kappa$  direction (the curvature is nevertheless horizontal). Indeed, we know that when we diagonalise, non-trivial  $T$ -bundles arise; and from the discussion in appendix A we know that these are torsion bundles.

Consider  $G = SU(2)$  and  $T = U(1)$ : a line bundle  $L$  on  $\Sigma_g$  has first Chern class  $c_1(L) = r[\omega]$ , so that  $\pi^*(L)$  has first Chern class

$$c_1(\pi^*(L)) = r[\pi^*(\omega)] = \frac{r}{p}[d\kappa]. \tag{5.1}$$

We thus see that the pull-back connection may be taken to be

$$A = 2\pi \frac{r}{p} \kappa, \tag{5.2}$$

which, as announced, lives in the  $\kappa$ -direction. It is perhaps somewhat surprising that, even though we have split off the part of the  $G$ -connection in the direction of  $\kappa$ , we are forced to reintroduce an (albeit non-dynamical) component in that direction upon diagonalisation.

This connection has holonomy in the  $S^1$  direction of  $M_{(g,p)}$ ,

$$\exp\left(i \oint A\right) = \exp\left(2\pi i \frac{r}{p}\right) \in \mathbb{Z}_p \tag{5.3}$$

and captures the torsion. The curvature 2-form  $F_H$  appearing in (4.6, 4.7) is then

$$F_H = dA_H + dA = dA_H + 2\pi \frac{r}{p} d\kappa \tag{5.4}$$

and the path integral should include a summation over  $r = 0, \dots, p - 1$ .

This argument generalises to higher rank. Normalising the component fields by expanding  $\phi$  and  $A_H$  in a basis of simple roots,

$$\phi = \sum_{i=1}^{\text{rk}} \phi^i \alpha_i, \quad A_H = \sum_{i=1}^{\text{rk}} A^i \alpha_i, \tag{5.5}$$

$F_H$  in (4.5) has the form

$$F_H = \sum_{i=1}^{\text{rk}} \left( dA_H^i + 2\pi \frac{r^i}{p} d\kappa \right) \alpha_i. \tag{5.6}$$

**Suming over bundles and a symmetry.** Our task is to sum over all allowed torus bundles, that is all torus bundles of finite order, on  $M$ . We should therefore sum over all allowed values of  $r^i = 0, \dots, p-1$  of (5.6). But how does the path integral (4.5) know that  $r^i = 0$  is the same as  $r^i = p$ ?

Note that shifting the  $r^i$  by multiples of  $p$ ,  $r^i \rightarrow r^i + p\gamma^i$ ,  $\gamma^i \in \mathbb{Z}$  is tantamount to shifting  $F_H$  by an element  $2\pi\gamma = 2\pi\gamma^i\alpha_i$  of the integral lattice  $I = 2\pi\mathbb{Z}[\alpha_i]$  of  $G$ . Thus consider the transformation

$$F_H \rightarrow F_H + 2\pi d\kappa \gamma \quad \phi \rightarrow \phi - 2\pi\gamma. \tag{5.7}$$

We claim that this is an invariance of the path integral (4.5). Indeed, even though the exponent is not invariant, it changes by

$$-i\pi(k + c_{\mathfrak{g}})p \sum C_{mn}\gamma^m\gamma^n. \tag{5.8}$$

Here  $C_{mn}$  is the Cartan matrix

$$C_{mn} = \text{Tr } \alpha_m \alpha_n. \tag{5.9}$$

Now  $C_{mn}$  is a symmetric integral matrix with even diagonal entries and consequently  $\sum C_{mn}\gamma^m\gamma^n$  is an even integer. Thus the phase (5.8) is  $2\pi it$  for some integer  $t$  and the exponential is invariant.

The Ray-Singer torsion term is also invariant under these transformations so they represent a symmetry of the theory at hand. This is consistent with the fact that we should only sum over the torsion classes, the symmetry guaranteeing that the result does not depend on the representative.

It is worth noting that, for  $p \neq 0$ , the symmetry (5.7) is not a symmetry of the original theory. Rather it reflects an ambiguity in our description of functions on  $M$  as sections of line bundles on  $\Sigma_g$ . Indeed, as we already discussed in section 2, the pull-back of the  $p$ -th power of any line bundle on  $\Sigma_g$  to  $M_{(g,p)}$  is trivial. Hence upon pull-back sections of a line bundle  $L$  on  $\Sigma_g$  are indistinguishable from sections of  $L \otimes \mathcal{L}^p$  for some line bundle  $\mathcal{L}$ . This is the origin of the ambiguity, which thus consistently appears as a symmetry of the theory.

This should be contrasted with what happens for  $p = 0$ : in that case  $d\kappa = 0$  and the background connection  $A$  has no  $\kappa$ -component. Nevertheless the symmetry  $\phi \rightarrow \phi + 2\pi\gamma$  exists as it is part of the original gauge symmetry (large Abelian gauge transformations wrapping around the  $S^1$ ). This gauge symmetry leads to  $\phi$  being a compact scalar taking values in  $\mathfrak{t}/I$ . While compactness of  $\phi$  is not required by the gauge symmetries for  $p \neq 0$ , we will see shortly that the symmetry (5.7) can nevertheless be used to provide an alternative description of the same theory in terms of a compact scalar (thus establishing the equivalence with the  $q$ -deformed Yang-Mills model of [7]).

**Reduction to finite dimensional integrals.** We had already argued at the end of section 3 that ultimately only constant  $\phi$  contribute to the path integral. To see this more explicitly, consider the  $\int \phi dA_H^t$  part of the action (4.7). The integral over  $A_H^t$  imposes the

required condition

$$d\phi^i = 0. \tag{5.10}$$

More precisely, as we have already explained, the harmonic parts of  $A_H^t$  do not enter in the action but just lead to an overall normalisation of the path integral. The exact parts of the gauge field do not appear either, but these are the components which are gauge degrees of freedom (the residual  $U(1)^{\text{rk}}$  gauge symmetry). That only leaves the co-exact piece of the gauge field and integration over these components imposes that the  $\phi^i$  are constant.

With  $\phi$  constant, the partition function (4.5) reduces to the finite-dimensional integral

$$Z_k[M_{(g,p)}, G] \sim e^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}_p^{\text{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \tag{5.11}$$

In particular, this formula reproduces (1.4) directly (for a judicious choice of normalisation) when we restrict to the Lens spaces  $L(p, 1)$ , take the group to be  $G = SU(2)$ , and deform the integration contour appropriately (there are no residues to worry about).

At this point we also refer back to the discussion in the Introduction where we compared and contrasted the general structure (1.3) of the formulae obtained from non-Abelian localisation with the structure (1.5) we found above that follows from diagonalisation and integrating out all the 2-dimensional gauge fields.

## 6. Compact versus non-compact Yang-Mills and interpretations

In this section we will look at what the solutions of these theories implies from the physics and mathematics points of view. Our starting point is that we can sum over all line bundles and declare the shift symmetry (5.7) to be a symmetry of the resulting Abelian theory (even though it is not a symmetry of the original theory).

Then given that we have this symmetry we should gauge fix it. One way to do that is to declare that we only sum over the required range of  $r$ . An alternative gauge fixing would be to compactify  $\phi$  as is done in [7]. Surprisingly enough the ‘parent’ theory (that is the one still to be gauge fixed) is just ordinary non-compact Yang-Mills theory on  $\Sigma_g$  with a gauge invariant observable inserted!

We first show directly that the two gauge fixed partition functions agree so that they indeed arise from one ‘parent’ partition function. For any function  $f(\phi)$  on  $\mathfrak{t}$ , invariant under shifts in the integral lattice  $I$ , and any integer  $q$  (for us  $q = (k + c_{\mathfrak{g}})$ ), let

$$Z_{q,p}(f) = \sum_{r \in \mathbb{Z}_p^{\text{rk}}} \int_{\mathfrak{t}} f(\phi) \exp i \frac{q}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \tag{6.1}$$

be a model of the partition function (5.11), and denote by  $\tilde{Z}_{q,p}(f)$  the analogous partition function for a compact scalar,

$$\tilde{Z}_{q,p}(f) = \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}/I} f(\phi) \exp i \frac{q}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi). \tag{6.2}$$

We now establish the identity

$$Z_{q,p}(f) = \tilde{Z}_{q,p}(f). \tag{6.3}$$

To that end we note that the integral over  $\mathfrak{t}$  is the same as integrating each lattice cell and then summing over the cells. However, one can shift any cell to a given one by an element of  $I$ . The integral over the given cell is then an integral over  $T = \mathfrak{t}/I$ .  $f(\phi)$  is, by hypothesis, invariant under such shifts. The exponent in (6.1), however, is not. Taking into account the shift one has

$$Z_{q,p}(f) = \sum_{s \in \mathbb{Z}_p^{\mathbf{rk}}} \sum_{n \in \mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}/I} f(\phi) \exp i \frac{q}{4\pi} \text{Tr} (p\phi^2 + 4(s + pn)\pi\phi) \tag{6.4}$$

As one sums over  $n$  and  $s$  the combination

$$r = np + s \tag{6.5}$$

with  $n \in \mathbb{Z}^{\mathbf{rk}}$  and  $s \in \mathbb{Z}_p^{\mathbf{rk}}$ , covers  $\mathbb{Z}^{\mathbf{rk}}$ . This establishes (6.3).

In particular, therefore, an equivalent way of writing the partition function (5.11) is

$$Z_k[M_{(g,p)}, G] \sim e^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}/I} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p\phi^2 + 4\pi r\phi) \tag{6.6}$$

This formula has been obtained at the level of the constant modes. But it shows us that the field theory which had a non-compact scalar  $\phi(x)$  and a finite sum over torsion bundles is equal to the field theory written down in [7], with a compact scalar and a  $\phi^2$ -term. Written this way the theory only makes sense with the infinite sum over bundles.

Now we turn to the theory which is to be gauge fixed to give either of the equivalent partition functions above. The parent theory is simply

$$W_{q,p}(f) = V^{-1} \sum_{r \in \mathbb{Z}^{\mathbf{rk}}} \int_{\mathfrak{t}} f(\phi) \exp i \frac{q}{4\pi} \text{Tr} (p\phi^2 + 4\pi r\phi) \tag{6.7}$$

where  $V = \text{Vol}(p\mathbb{Z}^{\mathbf{rk}})$  is the ‘gauge volume’. Fixing the gauge by passing from  $r \in \mathbb{Z}^{\mathbf{rk}}$  to  $r \in \mathbb{Z}_p^{\mathbf{rk}}$  one recovers  $\tilde{Z}_{q,p}(f)$ . Alternatively, writing  $r$  once more as in (6.5) and shifting  $\phi$  by the integral lattice elements to eliminate the  $n$ -dependence in the action one finds  $Z_{q,p}(f)$ .

**How compact is compact Yang-Mills?** Apart from compactness one of the main difference between the abelianised Chern-Simons theory on  $M_{(g,p)}$ , when finally reduced to a theory on  $\Sigma_g$ , and abelianised Yang-Mills theory on  $\Sigma_g$  is that the measure for the former involves  $\det(1 - e^{\text{ad } \phi})$  (related to the Weyl integral formula for Lie groups) and  $\det(\text{ad } \phi)$  (arising from the Weyl integral formula for Lie algebras) for the latter, both raised to the power  $\Sigma_g/2$ .

These two measures are related by the Jacobian of the exponential map on Lie groups, given (upon restriction to  $\mathfrak{t}$ ) by the function

$$j_{\mathfrak{g}}(\phi) = \frac{\det_{\mathfrak{t}}(1 - e^{\text{ad } \phi})}{\det_{\mathfrak{t}}(\text{ad } \phi)} \tag{6.8}$$

which makes an appearance in the study of coadjoint orbits and equivariant localisation — see e.g. [18].

Another difference rests in the normalisation of the action. Following the  $\phi F_A$  terms around we see that there is an extra factor of  $q$  in front of the Chern-Simons action compared to the Yang-Mills action in the normalisation adopted in [2]. This can be compensated for in the Yang-Mills theory if we send  $\phi \rightarrow q\phi$ . With this change understood, if we consider the expectation value of

$$j_{\mathfrak{g}}(\phi)^{\chi(\Sigma_g)/2} \tag{6.9}$$

in Yang-Mills theory on  $\Sigma_g$ , with a very particular value for the area, then on Abelianisation [2] we would arrive at (6.7) with  $f(\phi)$  given by the Ray-Singer torsion raised to half the Euler characteristic. Hence one is, up to possibly some normalisation factors, evaluating the Chern-Simons path integral on  $M_{(g,p)}$ .

What we have just exhibited is that the partition function and gauge invariant observables in compact Yang-Mills can be equally thought of as arising from non-compact Yang Mills theory but with the insertion of the operator (6.9).

**Cohomology of Yang-Mills connections on  $\Sigma_g$ .** Witten [5] has shown how to pass from the cohomological Yang-Mills theory on  $\Sigma_g$  to the physical theory. In this way pairings of the cohomology ring on the moduli space of flat connections on  $\Sigma_g$  can be extracted from the physical partition function. Higher critical points of the Yang-Mills action contribute to the path integral as well but these are exponentially suppressed as the area goes to zero.

In the present situation we cannot let the area go to zero since it is linearly related to  $p$ . Hence the contribution to the path integral, from the point of view of non-Abelian localisation [5, 4], includes components of the moduli space of Yang-Mills connections which are not flat.

We have already established that Chern-Simons theory on  $M_{(g,p)}$  is equivalent to Yang-Mills theory on  $\Sigma_g$  for a particular area. Consequently we now have that the Chern-Simons partition function yields certain intersection pairings on the moduli space of Yang-Mills connections on  $\Sigma_g$ . One can obtain precise formulae relating the two as has been done in [4].

**The Weyl group.** The formulae above are invariant under the action of the Weyl group  $W$  which is part of the original gauge group. We thus need to divide by the “volume” (or mod out by the action) of  $W$ .

For  $SU(2)$  the Weyl group is  $\mathbb{Z}_2$  and the non-trivial element sends the connection  $A$  to  $-A$ . As we emphasised previously it is as if we are working with a connection that is of the form

$$A = A_H + \phi \kappa + 2\pi \frac{r}{p} \kappa \tag{6.10}$$

and given the range of  $r$  in (5.11) the Weyl transformation corresponds to

$$A_H \rightarrow -A_H, \quad \phi \rightarrow -\phi - 2\pi, \quad r \rightarrow p - r. \tag{6.11}$$

When we abelianise on  $\Sigma_g$  we are essentially splitting the  $SU(2)$  bundle in a direct sum of lines

$$\text{ad}_{\mathbb{C}}(P_{SU(2)}) = \mathcal{O}(-2n) \oplus \mathcal{O} \oplus \mathcal{O}(2n) \tag{6.12}$$

and the Weyl group acts to exchange  $\mathcal{O}(-2n)$  and  $\mathcal{O}(2n)$ . The Chern classes  $n$  and  $-n$  do not necessarily agree mod  $p$  and so do not correspond to the same torsion class on  $M_{(g,p)}$ . Rather  $-n$  for  $n \geq 0$  is the same as  $p - n$  and explains the last transformation in (6.11).

On the other hand, once one has transformed to (6.6) one sees that the Weyl group acts by  $\phi \rightarrow -\phi$  and  $r \rightarrow -r$ . Indeed if one keeps track of the field redefinitions then these transformations agree with (6.11). The formula that we will use mostly in the following is

$$Z_k[M_{(g,p)}, G] = \Lambda e^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}/\Gamma^W} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (6.13)$$

where  $\Gamma^W = I \rtimes W$  is the affine Weyl group and  $\Lambda$  is a real normalisation constant that remains to be determined.

Notice that the Ray-Singer torsion has zeros at the boundary of the Weyl chamber, which means that for genus  $g > 1$  the integrals (5.11, 6.13) diverge. As already indicated in [1] the way around this is to regularise by giving a small mass term to the connection, while preserving the residual  $U(1)^{\text{rk}}$  invariance. It is easier to incorporate this in the form (6.13) since this amounts to not including the boundaries of the Weyl chamber. As we will see presently the contributions to the path integral are at discrete points and so the regularisation renders the integrals finite.

### 7. The partition function on $M_{(g,p)}$

We have thus found the following equivalent finite-dimensional integral expressions for the partition function on  $M_{(g,p)}$ ,

$$Z_k[M_{(g,p)}, G] = \Lambda \frac{e^{4\pi i p \Phi_0}}{|W|} \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (7.1)$$

$$= \Lambda \frac{e^{4\pi i p \Phi_0}}{|W|} \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}/I} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (7.2)$$

$$= \Lambda e^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}^{\text{rk}}} \int_{\mathfrak{t}/\Gamma^W} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} (p \phi^2 + 4\pi r \phi) \quad (7.3)$$

with  $\Lambda$  real.

One could have kept tabs on most of the normalisation factors. It is more convenient, however, to give a prescription for fixing  $\Lambda$ . One takes  $\Lambda$  to be independent of  $p$  and chosen so that for  $p = 0$  (7.3) agrees with the partition function obtained on  $\Sigma_g \times S^1$  in [1], that is, it reproduces the Verlinde formula for dimension of the space of conformal blocks on  $\Sigma_g$ .<sup>3</sup>

Once one has fixed the normalisation as we have then the partition function on  $M_{(g,p)}$  is the same as the expectation value of an operator in the theory on  $\Sigma_g \times S^1$ ,

$$Z_k[M_{(g,p)}, G] = e^{4\pi i p \Phi_0} \langle e^{i p \frac{(k+c_{\mathfrak{g}})}{4\pi} \text{Tr} \phi^2} \rangle_{\Sigma_g \times S^1} \quad (7.4)$$

---

<sup>3</sup>For  $G = SU(2)$  this fixes  $\Lambda = (2(k+2))^g / 2\pi$ .

where on the right hand side  $\phi$  is the constant mode of the gauge field in the  $S^1$  direction and we have already imposed that  $\phi$  is in the Cartan sub-algebra (and, as discussed before,  $\phi$  is compact in the  $G/G$ -model). The difficulty in finding the correct non-gauge fixed version of the operator  $\exp i \frac{(k+c_{\mathfrak{g}})}{4\pi} \text{Tr} \phi^2$  is at the heart of the problem of developing a  $G/G$  type model corresponding to  $M_{(g,p)}$ .

Nevertheless, this is a remarkable (and remarkably simple) result. It states that manifolds of non-trivial Chern classes are simply created by insertions of the operator  $\exp i \frac{(k+c_{\mathfrak{g}})}{4\pi} \text{Tr} \phi^2$  in the path integral for the trivial bundle. That this should be so has first been argued by Vafa [6] in a rather different way. Here we have provided a path integral derivation of this fact.

As in [1], the sum over  $r \in \mathbb{Z}^{\text{rk}}$  imposes a  $\delta$ -function constraint on the field  $\phi$  to configurations that satisfy

$$\phi = \frac{2\pi(\lambda + \rho)}{k + c_{\mathfrak{g}}} \tag{7.5}$$

for some weight  $\lambda$ . Here  $\rho$  is the Weyl vector, one half the sum of the positive roots. In particular we have

$$\frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} \phi^2 = \frac{\pi}{(k + c_{\mathfrak{g}})} \text{Tr}((\lambda + \rho)^2) = -\frac{2\pi}{(k + c_{\mathfrak{g}})} \left( c_2(\lambda) + c_{\mathfrak{g}} \frac{\dim G}{24} \right) \tag{7.6}$$

where the quadratic Casimir is

$$c_2(\lambda) = -\frac{1}{2} \text{Tr} (\lambda^2 + 2\rho\lambda) \tag{7.7}$$

and we have made use of the Freudenthal - de Vriess formula

$$\text{Tr} \rho^2 = -c_{\mathfrak{g}} \frac{\dim G}{12}. \tag{7.8}$$

Combining this with the overall phase we have that

$$\frac{k + c_{\mathfrak{g}}}{4\pi} \text{Tr} \phi^2 + 4\pi\Phi_0 = -2\pi\Phi(\lambda) \tag{7.9}$$

where

$$\Phi(\lambda) = \frac{1}{(k + c_{\mathfrak{g}})} \left( c_2(\lambda) - \dim G \frac{k}{24} \right). \tag{7.10}$$

Hence the  $T$  matrix

$$T_{\lambda\mu} = \delta_{\lambda\mu} T_{\lambda} \quad T_{\lambda} = \exp 2\pi i \Phi(\lambda) \tag{7.11}$$

appears naturally in the evaluation of the path integral.

The compactness of  $\phi$ ,  $\phi \in \mathfrak{t}/\Gamma^W$ , constrains the possible weights so that they are integrable highest weights at level  $k$ . Writing the partition function on  $\Sigma_g \times S^1$  as

$$Z_k[\Sigma_g \times S^1, G] = \sum_{\lambda} S_{0\lambda}^{2-2g} \tag{7.12}$$

where the sum is over integrable highest weights, with our normalisation one then has

$$Z_k[M_{(g,p)}, G] = \sum_{\lambda} S_{0\lambda}^{2-2g} T_{\lambda}^{-p}. \tag{7.13}$$

Note that both the  $T$  and  $S$  matrices are the standard modular matrices for the Wess-Zumino-Witten model.  $T$  is diagonal (in the Verlinde basis) and  $S$  is symmetric [19]. These mapping class group generators are subject to the relations

$$S^2 = \mathbb{I}, \quad (ST)^3 = \mathbb{I}. \tag{7.14}$$

We wish to interpret the results of performing the path integral in terms of the surgery prescription of Chern-Simons theory. In order to do that we need to introduce Wilson loops which is what we do next.

### 8. Surgery and the framing provided by the path integral

Since (7.13) is already reminiscent of a surgery formula we would expect that indeed

$$Z_k[M_{(g,p)}, G] = \sum_{\lambda} K_{0\lambda}^{(p)} Z_k[\Sigma_g \times S^1, G, R_{\lambda}] \tag{8.1}$$

for some mapping class group element  $K^{(p)}$ . In this and subsequent formulae  $R_{\mu}$  is a Wilson loop in the vertical direction in the representation  $\mu$ . Since it is the integrals over  $A_H^t$  which ensure that  $\phi$  is constant we see that the inclusion of Wilson lines into the path integral does not change this. So we have not indicated any base point dependence as at the end of the day there is no such dependence. We know from [1] that

$$Z_k[\Sigma_g \times S^1, G, R_{\lambda}] = \sum_{\sigma} S_{0\sigma}^{1-2g} S_{\lambda\sigma} \tag{8.2}$$

and so comparing with (7.13) suggests that

$$K^{(p)} = ST^{-p}S \tag{8.3}$$

which allows us to write  $K^{(p)} = K^p$  where

$$K = ST^{-1}S. \tag{8.4}$$

The canonical choice for obtaining  $S^3$  from a surgery on  $S^2 \times S^1$  is to identify up to the action of  $S$ , however, quite generally the surgery

$$K = T^m ST^n \tag{8.5}$$

yields  $S^3$  with  $m$  and  $n$  arbitrary integers. Different values of  $m$  and  $n$  correspond to different framings of  $S^3$ . The path integral calculation will correspond to one of the possible matrices in (8.5), that is it will correspond to a definite value for  $m$  and  $n$  and hence to a particular framing of  $S^3$ .

Witten [19] has shown that in the canonical framing one has  $K = S$ .<sup>4</sup> By making use of the second relation in (7.14) we see that we can write  $K$  as

$$K = TST \tag{8.6}$$

suggesting that  $m = n = 1$  so that we are not working in the canonical framing if our identification (8.3) is correct.

As an aside we note that one could have used the non-compact representation (7.1) of the partition function rather than (7.3) to calculate the Hopf link. We show in appendix C that for  $G = SU(2)$  this leads directly to  $K = TST$  without invoking (7.14). Since we have calculated the same quantity  $K$  in two different ways, we thus have a path integral derivation of the relation  $TST = ST^{-1}S$ .

Consider now the more general surgery formulae involving two vertical Wilson lines

$$Z_k[M_{(g,p)}, G, R_\mu R_\nu] = \sum_\lambda K_{\mu\lambda}^{(p)} Z_k[\Sigma_g \times S^1, G, R_\lambda R_\nu] \tag{8.7}$$

(the vertical Wilson line  $R_\nu$  is not in the tubular neighbourhood where the surgery takes place but  $R_\mu$  is). In particular for  $g = 0, p = 1$  one is calculating the expectation value of the Hopf link since, on noting that  $Z_k[S^2 \times S^1, G, R_\lambda R_\nu] = \delta_{\lambda\nu}$ , one finds

$$K_{\mu\nu} = Z_k[S^3, G, R_\mu R_\nu]. \tag{8.8}$$

The left hand side of (8.7) is easy to determine given the constraint (7.5) implied in (7.3),

$$Z_k[M_{(g,p)}, G, R_\mu R_\nu] = \sum_\lambda S_{0\lambda}^{-2g} S_{\lambda\mu} S_{\lambda\nu} T_\lambda^{-p}. \tag{8.9}$$

Comparing with (8.7) we see that indeed  $K^{(p)}$  is as in (8.3) and that  $K$  is given by (8.4) and consequently (8.6). Indeed one obtains the formula for  $K$  directly by setting  $g = 0, p = 1$  in (8.9).

We read off from (8.8) and (8.6) that we have framed the vertical knots by +1 units from the canonical framing of the knot. This is related to the -2 unit of framing for  $S^3$  that we are away from the canonical framing as determined by Beasley and Witten [4]. In the current setting we see this framing of  $S^3$  on setting  $\mu = \nu = 0$ .

Beasley and Witten [4] determine the change of framing in a related but somewhat different manner. The advantage of our approach is that it allowed us to identify the element,  $K^{(p)}$ , of the mapping class group with which one builds non-trivial bundles and which, at the same time, incorporates the change of framing.

The operator  $K = TST$  was also identified in [20] as the operator that generates the insertion (1.9) of the Chern class in the matrix model. On the other hand, comparing with [14], we note that q-deformed Yang-Mills theory, as defined there, differs from our definition by the (insignificant) change  $p \rightarrow -p$ , accounting for the difference

$$K = ST^{-1}S \rightarrow K' = STS = T^{-1}ST^{-1}. \tag{8.10}$$

Indeed, if one uses  $K'$  instead of  $K$  in the surgery prescription, one finds that the partition function is given by (7.13) with  $p \rightarrow -p$ .

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<sup>4</sup>Note that with this choice of  $K$  one cannot possibly have  $K^{(p)} = K^p$  since  $S^2 = 1$ .

## 9. Generalisations

There are various generalisations that are possible. These include the choice of other gauge groups. We have not considered either non-compact gauge groups or compact but not simply connected gauge groups thus far in this paper. Also one may ask about other 3-manifolds, do our considerations above apply?

The steps involved in solving the theory on  $U(1)$  bundles were:

1. Decomposing the gauge field as (2.14).
2. Fixing the gauge (2.21).
3. Diagonalising (2.22)
4. Pushing the calculations down to the base  $\Sigma_g$ .

Step 1 is always possible since one can introduce a contact structure on any 3-manifold and this automatically gives us the required decomposition. However,  $\kappa$  may not be globally defined in which case  $\phi$  is not globally defined either but is correctly thought of as a section of some real line bundle.  $\kappa\phi$  is globally well defined.

Step 2 now follows but one must ensure that the derivatives in the gauge fixing condition are covariant derivatives with respect to the real line bundle.

Step 3 is not straightforward in general. Compact but not simply connected groups pose a problem. While we can indeed abelianise gauge theories for non-simply connected groups on a Riemann surface (see [15] for a general discussion and [21] for concrete applications), there are other obstructions to diagonalisation in 3 dimensions and above [15]. Non-compact groups are already an issue on Riemann surfaces. The problem here is that there may be different maximal tori to which one conjugates. Likewise, there is a problem in extending Abelianisation to other 3-manifolds again due to obstructions beyond the ones we have already considered. However, there is the hope that the technique will be applicable in the general Seifert fibred case, since this amounts to establishing a diagonalisation for 2-dimensional orbifolds, but that is still to be shown.

Step 4 is certainly a technical convenience but may not be essential, otherwise for a general 3-manifold we would not know how to proceed.

In the above discussion it is step 3 which is the most difficult to mimic in other cases. However, for special non-compact groups one may apply all of the machinery. We finish with an example of this.

3-dimensional BF-theories, topological gauge theories with action

$$S_{BF} = \int_M \text{Tr } B \wedge F_A, \tag{9.1}$$

where  $B$  is a Lie algebra valued 1-form, can be interpreted as Chern-Simons theories with a non-compact gauge group usually denoted by  $IG$  or  $TG \sim G \times \mathfrak{g}$  [22, 23]. There is no sufficiently well understood surgery/CFT prescription for calculating the partition function of pure BF theory and a direct path integral evaluation of the partition function is desirable.

On 3-manifolds which have isolated flat connections, the BF path integral is formally a sum of the Ray-Singer torsions of each of the flat connections. When the flat connections are not isolated, there are  $B$  zero modes and one gets an integral over the tangent bundle of the moduli space of flat connections. To avoid that we fix our attention on the Lens spaces  $L(p, 1)$ . One can now use the analysis of the previous sections to get explicit formulae.

Apart from the usual gauge symmetry, the action (9.1) has the invariance

$$B \rightarrow B + d_A f \tag{9.2}$$

To perform calculations we decompose  $A$  as before and  $B$  as

$$B = B_H + \kappa \lambda. \tag{9.3}$$

With this decomposition the action (9.1) becomes

$$\int_M \text{Tr} (B_H \wedge d_\phi A_H + B_H \wedge d(\kappa \phi) + \lambda \kappa \wedge d(\kappa \phi) + \lambda \kappa \wedge d_\phi A_H + \lambda \kappa \wedge A_H \wedge A_H) \tag{9.4}$$

We impose the gauge conditions (2.21) and (2.22) on  $\phi$  and similar conditions on  $\lambda$ ,

$$L_K \lambda = 0, \quad \lambda^\natural = 0. \tag{9.5}$$

Now the path integral over invariant  $A_H$  implies that the  $\lambda$  are constant and the integral over invariant  $B_H$  implies that the  $\phi$  are constant. The other integrals together with the ghost terms yield the Ray-Singer torsion, so (for  $G = SU(2)$  for simplicity)

$$Z_{BF}[L(p, 1), G = SU(2)] = \sum_{r=0}^{p-1} \int d\lambda d\phi T_{S^1}(\phi)^2 \exp(ip \text{Tr} \lambda \phi + ir \lambda) \tag{9.6}$$

The integral over  $\lambda$  gives us a delta function on  $\phi$  so that  $\exp \phi$  is a root of unity,

$$Z_{BF}[L(p, 1), SU(2)] = \sum_{r=0}^{p-1} T_{S^1}(2\pi ir/p)^2 = \sum_{r=0}^{p-1} T_{L(p,1)}(2\pi ir/p). \tag{9.7}$$

One can also evaluate the expectation values for Wilson loops in the fibre direction with and without the  $B$  field. We leave that for another occasion.

Finally, we expect that super  $BF$  theory can be solved in a similar way on Lens spaces, thus giving us the  $SU(n)$  Casson invariant in these cases.

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### A. Diagonalisation and torsion bundles on $M_{(g,p)}$

**Line bundles on  $M_{(g,p)}$  and  $\Sigma_g$  and the Gysin sequence.** The 3-manifolds  $M_{(g,p)}$  that we are interested in are principal  $U(1)$ -bundles over a 2-dimensional surface  $\Sigma_g$  of genus  $g$ ,  $U(1) \rightarrow M_{(g,p)} \xrightarrow{\pi} \Sigma_g$ . There is a first Chern class associated with this fibration which we denote by  $-p$ . Thinking of  $M_{(g,p)}$  as an  $SO(2)$  bundle this is the real Euler class, while the associated complex line bundle on  $\Sigma_g$  is  $\mathcal{O}(-p)$ . We will need to know various basic things about line bundles on  $M_{(g,p)}$  and their relations to line bundles on  $\Sigma_g$ .

To fix notions a bit we need some background. Given a vector field  $v$  on  $\Sigma_g$  we can lift it to  $M_{(g,p)}$  by demanding that its lift,  $\widehat{v}$ , satisfies  $\iota_{\widehat{v}}\kappa = 0$  (this picks out horizontal subspaces). In terms of the local coordinates (2.5) this means that  $\widehat{v} = (v, -a.v)$ . Also, in the standard manner,  $U(1)$ -equivariant complex valued functions  $f_m$  on  $M_{(g,p)}$ , with our normalisation of the  $U(1)$ -generator  $K$ , satisfy

$$L_K f_m = 2\pi i m f_m. \tag{A.1}$$

Covariant derivatives along  $v$  are just actions of the vector field  $\widehat{v}$  on equivariant functions on  $M_{(g,p)}$ ,

$$\widehat{v}.f_m = v^i \left( \partial_i - a_i \frac{\partial}{\partial \theta} \right) f_m = v^i (\partial_i - 2\pi i m a_i) f_m. \tag{A.2}$$

Consequently the equivariant functions  $f_m$  are in one-to-one correspondence with sections of the corresponding associated line bundle, i.e. with sections of  $\mathcal{O}(mp)$  given that the connection  $a$  is a connection on  $\mathcal{O}(-p)$  (2.11).

Moreover, the pull-back  $\pi^*\mathcal{O}(p)$  from  $\Sigma_g$  to  $M_{(g,p)}$  is (tautologically) trivial. Hence

$$(\pi^*\mathcal{O}(m))^{\otimes p} \approx \pi^*\mathcal{O}(mp) \approx (\pi^*\mathcal{O}(p))^{\otimes m} \tag{A.3}$$

is also trivial for any  $m \in \mathbb{Z}$ . It follows that the  $p$ -th power of the pull-back to  $M_{(g,p)}$  of any line bundle  $L$  on  $\Sigma_g$  is trivial,

$$(\pi^*L)^{\otimes p} \approx M_{(g,p)} \times \mathbb{C}. \tag{A.4}$$

Since line bundles on  $M_{(g,p)}$  are classified by  $H^2(M_{(g,p)}, \mathbb{Z})$ ,  $\pi^*L$  must be such that

$$pc_1(\pi^*L) = 0 \in H^2(M_{(g,p)}, \mathbb{Z}), \tag{A.5}$$

i.e. that such line bundles correspond to  $p$ -torsion.

We will now determine  $H^2(M_{(g,p)}, \mathbb{Z})$ , show that it indeed contains torsion  $\mathbb{Z}_p$ , and that all such torsion bundles arise via pull-back from  $\Sigma_g$ . We will address these issues using the Gysin sequence for sphere bundles.

Given a sphere bundle  $M$  with fibre  $F = S^m$  over a manifold  $B$  one has the long exact sequence,

$$\dots \rightarrow H^n(M) \xrightarrow{\pi_*} H^{n-m}(B) \xrightarrow{\wedge^e} H^{n+1}(B) \xrightarrow{\pi^*} H^{n+1}(M) \rightarrow \dots \tag{A.6}$$

where  $\pi_*$  is integration along the fibre (push down),  $\wedge^e$  is wedging with respect to the Euler class and  $\pi^*$  is the usual pull back. The coefficients can be the integers.

In our case  $m = 1$ ,  $F = S^1$ ,  $B = \Sigma_g$  and the Euler class is the first Chern class of the  $U(1)$  bundle over  $\Sigma_g$ . We have, therefore,

$$0 \rightarrow H^1(\Sigma_g, \mathbb{Z}) \xrightarrow{\pi^*} H^1(M, \mathbb{Z}) \xrightarrow{\pi_*} H^0(\Sigma_g, \mathbb{Z}) \xrightarrow{\wedge^{c_1}} H^2(\Sigma_g, \mathbb{Z}) \xrightarrow{\pi^*} H^2(M, \mathbb{Z}) \xrightarrow{\pi_*} H^1(\Sigma_g, \mathbb{Z}) \rightarrow 0. \quad (\text{A.7})$$

We see from the sequence that if  $c_1$  is not trivial, as is the case for the manifolds under consideration, then

$$H^2(\Sigma_g, \mathbb{Z}) \xrightarrow{\pi^*} H^2(M, \mathbb{Z}) \quad (\text{A.8})$$

is surjective only when  $\Sigma_g$  is the 2-sphere  $S^2$ , i.e. only in that case do all line bundles on  $M_{(g,p)}$  arise from line bundles on  $\Sigma_g$  via pull-back.

One can also read off, from the sequence, that while  $H^0(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$  and  $H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$  are isomorphic, that the map between the two is via multiplication by the integer  $p = c_1[\Sigma_g]$ . As long as  $p \neq 0$  this map has 0 kernel and so the image of  $H^1(M, \mathbb{Z})$  under  $\pi_*$  is 0, so that we deduce that

$$H^1(M, \mathbb{Z}) \equiv H^1(\Sigma_g, \mathbb{Z}). \quad (\text{A.9})$$

We can, therefore, re-write the sequence as

$$0 \rightarrow H^0(\Sigma_g, \mathbb{Z}) \xrightarrow{\wedge^{c_1}} H^2(\Sigma_g, \mathbb{Z}) \xrightarrow{\pi^*} H^2(M, \mathbb{Z}) \xrightarrow{\pi_*} H^1(\Sigma_g, \mathbb{Z}) \rightarrow 0 \quad (\text{A.10})$$

or

$$0 \rightarrow \mathbb{Z}_p \xrightarrow{\pi^*} H^2(M, \mathbb{Z}) \xrightarrow{\pi_*} H^1(\Sigma_g, \mathbb{Z}) \rightarrow 0. \quad (\text{A.11})$$

Essentially we learn that (as sets)

$$H^2(M_{(g,p)}, \mathbb{Z}) = H^1(\Sigma_g, \mathbb{Z}) \oplus H^2(\Sigma_g, \mathbb{Z})/p\mathbb{Z} = H^1(\Sigma_g, \mathbb{Z}) \oplus \mathbb{Z}_p, \quad (\text{A.12})$$

with the second summand arising from pull-back of classes (and hence line bundles) from the base. Consequently, finite order bundles on  $M_{(g,p)}$  are classified by  $\mathbb{Z}_p$ , pull-backs of bundles from the base  $\Sigma_g$  are of finite order  $p$ , and all such finite order line bundles on  $M_{(g,p)}$  arise as pull backs of bundles from the base.

For  $p = 0$ , i.e. the trivial bundle  $\Sigma_g \times S^1$ , the Gysin sequence gives back the Künneth formula,

$$H^2(M, \mathbb{Z}) = H^2(\Sigma_g, \mathbb{Z}) \oplus H^1(\Sigma_g, \mathbb{Z}) \cdot H^1(S^1, \mathbb{Z}) = H^2(\Sigma_g, \mathbb{Z}) \oplus H^1(\Sigma_g, \mathbb{Z}), \quad (\text{A.13})$$

there is no torsion, and line bundles on the base and their pull-backs are both classified by  $H^2(\Sigma_g, \mathbb{Z})$ .

**Diagonalisation.** Let  $P_G$  be a principal  $G$  bundle over  $\Sigma_g$  and denote by  $\pi_V : V_{\mathfrak{g}} \rightarrow \Sigma_g$  the associated **ad** vector bundle. For  $G$  simply connected,  $P_G$  and  $V_{\mathfrak{g}}$  are necessarily trivial(isable). Let  $\varphi$  be a section of  $V_{\mathfrak{g}}$ . As explained in [1, 15], one may conjugate  $\varphi$  (by vertical automorphisms  $g \in \text{Map}(M, G)$  of  $V_{\mathfrak{g}}$ ) such that it takes values in the Cartan subalgebra  $\mathfrak{t}$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . As such,  $\varphi$  becomes a section of a  $\mathfrak{t}$ -bundle  $V_{\mathfrak{t}}$  over  $\Sigma_g$  which need however not be trivial.  $\mathfrak{t}$ -bundles are classified by  $H^2(\Sigma_g, \mathbb{Z}^{\text{rk}}) \sim \mathbb{Z}^{\text{rk}}$ , and all  $\mathfrak{t}$ -bundles are engendered in this way. In other words, all non-trivial  $\mathfrak{t}$ - or  $T$ -bundles arise as obstruction bundles to diagonalisation in this case.

In general, diagonalisation and the obstructions to diagonalisation are not well understood in more than 2 dimensions (see the discussion in [15]). However, all we need is (see section 2) the diagonalisation of  $U(1)$ -invariant  $\mathfrak{g}$ -valued sections on  $M_{(g,p)}$  and this is straightforward.

First of all, we can pull back  $V_{\mathfrak{g}}$  to  $M_{(g,p)}$ ,  $\pi^*V_{\mathfrak{g}} \approx M_{(g,p)} \times \mathfrak{g}$ . The diagonalising maps can be pulled back to  $M_{(g,p)}$  as well and so we have diagonalised  $\pi^*\varphi$  on  $M_{(g,p)}$ . If the diagonalised  $\varphi$  was a section of  $V_{\mathfrak{t}}$ ,  $\pi^*\varphi$  is a section of  $\pi^*V_{\mathfrak{t}}$ . Thus the pull-backs of  $\mathfrak{t}$ -bundles on  $\Sigma_g$  to  $M_{(g,p)}$  are the obstructions to diagonalising sections of pull-back bundles.

Secondly, sections of pull-back bundles are in one-to-one correspondence with  $U(1)$ -invariant sections of bundles on  $M_{(g,p)}$ : clearly by definition  $\pi^*\varphi$  is  $U(1)$ -invariant, and conversely any  $U(1)$ -invariant section is such that its value depends on  $m \in M_{(g,p)}$  only via  $\pi(m) \in \Sigma_g$ , and hence it can be identified with the section of a line bundle on the base.

We can thus conclude that the problem of diagonalising  $U(1)$ -invariant  $\mathfrak{g}$ -valued sections of bundles on  $M_{(g,p)}$  can be reduced to the well-understood problem of diagonalisation on the base  $\Sigma_g$ . The diagonalised sections are sections of the pull-backs of the  $\mathfrak{t}$ -bundles  $V_{\mathfrak{t}}$  on  $\Sigma_g$ . As we have shown, these are precisely the torsion bundles on  $M_{(g,p)}$ . Thus the price of conjugating an invariant section into the Cartan subalgebra  $\mathfrak{t}$  is to “liberate” all these  $\mathfrak{t}$ -bundles of finite order on  $M_{(g,p)}$ . This accounts for the ubiquitous appearance of summations over  $\mathbb{Z}_p^{\text{rk}}$  in the equations of sections 5 to 7.

## B. Regularising the determinants

The path integrals that we encounter in the text are formally roots of determinants,

$$\int D\Phi \exp\left(i \int \Phi * Q \Phi\right) = \frac{1}{\sqrt{\text{Det } Q}}. \tag{B.1}$$

Because of the oscillatory nature of the path integral, we set, for an operator  $Q$ ,

$$\sqrt{\text{Det } Q} = \sqrt{|\text{Det } Q|} \exp \frac{+i\pi}{2} \eta(Q) \tag{B.2}$$

where

$$\eta(Q) = \frac{1}{2} \sum_{\lambda} \text{sign}(\lambda) \tag{B.3}$$

and  $\lambda$  are the eigenvalues of  $Q$  and the root is the positive root. The + sign in front of  $\eta(Q)$  in the phase appears here because the scalar product (trace) implicit in (B.1) is negative definite for anti-hermitian fields.

Both the absolute value and the phase of the determinants require regularisation. We regularise the absolute value and the phase (assuming that zero is not an eigenvalue) by setting

$$|\text{Det } Q|(s) = \exp \sum_{\lambda} e^{s\Delta} \ln |\lambda| \tag{B.4}$$

$$\eta(Q, s) = \frac{1}{2} \sum_{\lambda} \frac{\text{sign}(\lambda)}{|\lambda|^s} \exp s\Delta \tag{B.5}$$

for some appropriate operator  $\Delta$ .

**Some geometry.** As everything in sight respects the geometric  $U(1)$  structure on  $M$ , we decompose all fields with respect to this action. This means that, as in [4, section 5] we expand all the fields in eigenmodes of the Lie derivative  $L_K$ . So we set

$$A_H = \sum_{n=-\infty}^{\infty} A_n, \tag{B.6}$$

where the eigenmodes satisfy

$$L_K A_n = -2\pi i n A_n \quad \iota_K A_n = 0, \tag{B.7}$$

and likewise for the ghosts  $c$  and  $\bar{c}$ . By the discussion in appendix A, these eigenmodes can equivalently be regarded as sections of line bundles  $\mathcal{O}(-np)$  over  $\Sigma_g$  (which pull back to the trivial line bundle on  $M_{(g,p)}$ ).

Hence we have that

$$\Omega^0(M, \mathbb{C}) = \bigoplus_n \Omega^0(\Sigma_g, \mathcal{O}(-np)), \tag{B.8}$$

and on tensoring with the trivial bundles  $V_{\mathfrak{k}}$  below and  $\pi^*(V_{\mathfrak{k}}) = M \times \mathfrak{k}$  above we have

$$\Omega^0(M, \mathfrak{k}) = \bigoplus_n \Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}}). \tag{B.9}$$

A similar discussion shows that each mode  $n$  of a horizontal 1-form on  $M$  is one to one with a section on  $\Sigma_g$ , consequently one has

$$\Omega^1_H(M, \mathfrak{k}) = \bigoplus_n \Omega^1(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}}). \tag{B.10}$$

**Computing the ratio of determinants.** As explained in [1], ratios of determinants of the form of (4.2) almost cancel. They would cancel except for mismatches of harmonic modes. The ratio of determinants (4.2) is essentially

$$\prod_n (2\pi n + \text{ad } \phi)_{\mathfrak{k}}^{\dim \Omega^0(\Sigma_g, \mathcal{O}(-np)) - \frac{1}{2} \dim \Omega^1(\Sigma_g, \mathcal{O}(-np))} \tag{B.11}$$

The ratio may now be deduced by using the Hodge decomposition and an index theorem on  $\Sigma_g$ . We set

$$\Omega^1(\Sigma_g, \mathcal{O}(-np)) = \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}}) \oplus \Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}}), \tag{B.12}$$

with the complex structure on  $\Sigma_g$  chosen to be consistent with the standard orientation,  $*_{\Sigma} dz = -i dz$ .

Consider the determinant coming from the ghosts. This is

$$|\text{Det } i(\mathcal{L}_\phi)| \tag{B.13}$$

acting on  $\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})$  and the absolute value is there, since the ghost determinant should be a real volume (of the gauge group). Using our regularisation we re-write the ghost determinant as

$$\sqrt{\text{Det } i(\mathcal{L}_\phi)} \cdot \sqrt{\text{Det } i(-\mathcal{L}_\phi)} \tag{B.14}$$

the phases cancelling between the square roots since, by (B.5),  $\eta(Q, s) + \eta(-Q, s) = 0$ .

Upon decomposing the space of 1-forms as

$$\Omega^1(\Sigma_g, \mathcal{O}(-np)) = \Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np)) \oplus \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np)) \tag{B.15}$$

the ratio of determinants (4.2) that we wish to calculate becomes

$$\prod_{n=-\infty}^{\infty} \sqrt{\frac{\text{Det } (i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}}{\text{Det } (i\mathcal{L}_\phi)_{\Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}} \cdot \frac{\text{Det } (-i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}}{\text{Det } (-i\mathcal{L}_\phi)_{\Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}}}. \tag{B.16}$$

Since we are calculating ratios of determinants the traces in both (B.4) and (B.5) will involve the differences of traces on the spaces  $\Omega^*(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})$ . The eigenvalues are constants, so that the regularised traces can be evaluated directly, as  $s \rightarrow 0$ , on setting  $\Delta$  to be the appropriate Laplacians

$$[\text{Tr}_{\Omega^{(0,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})} - \text{Tr}_{\Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}] \exp s\Delta = \chi(\mathcal{O}(-np) \otimes V_{\mathfrak{k}}) \tag{B.17}$$

and

$$[\text{Tr}_{\Omega^{(0,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})} - \text{Tr}_{\Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_{\mathfrak{k}})}] \exp s\Delta = -\chi(K \otimes \mathcal{O}(-np) \otimes V_{\mathfrak{k}}), \tag{B.18}$$

where  $K$  is the canonical bundle of  $\Sigma_g$ . Note that

$$\begin{aligned} \chi(\mathcal{O}(-np) \otimes V_{\mathfrak{k}}) + \chi(K \otimes \mathcal{O}(-np) \otimes V_{\mathfrak{k}}) &= 2(c_1(\mathcal{O}(-np)) + c_1(V_{\mathfrak{k}})) \\ \chi(\mathcal{O}(-np) \otimes V_{\mathfrak{k}}) - \chi(K \otimes \mathcal{O}(-np) \otimes V_{\mathfrak{k}}) &= \chi(\Sigma_g). \end{aligned} \tag{B.19}$$

We decompose the complexified Lie algebra as  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$ . Denote the roots by  $\alpha$ . Furthermore we decompose  $\mathfrak{k}_{\mathbb{C}}$  in terms of root spaces and we write  $V_{\mathfrak{k}} = \oplus_{\alpha} V_{\alpha}$ .  $\text{ad } \phi$  acts on  $V_{\alpha}$  by multiplication by  $\alpha(\phi)$ . With our conventions  $i\alpha(\phi)$  is real. Covariant derivatives on the root space  $V_{\alpha}$  are  $d + \text{ad } A = d + \alpha(A)$ . Hence  $c_1(V_{\alpha}) = -\alpha(F_A)/2\pi i$ . Moreover, for  $X, Y \in \mathfrak{t}$  we have

$$\text{Tr } \text{ad}(X) \text{ad}(Y) = -2 \sum_{\alpha > 0} i\alpha(X) i\alpha(Y) = 2c_{\mathfrak{g}} \text{Tr } XY \tag{B.20}$$

with  $c_{\mathfrak{g}}$  the Coxeter number.

**The absolute value.** Set for any field  $\psi_{n\alpha} \in \Omega^*(\Sigma_g, \mathcal{O}(-np) \otimes V_\alpha)$ ,

$$i\mathcal{L}_\phi \psi_{n\alpha} = M_{n,\alpha} \psi_{n\alpha}, \quad M_{n,\alpha} = (2\pi n + i\alpha(\phi)) \quad (\text{B.21})$$

so that the determinants that we are interested in are essentially products of such  $M_{n,\alpha}$  over the roots,

$$\text{Det}(i\mathcal{L}_\phi)_{\Omega^*(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})} = \prod_\alpha \text{Det}_{\Omega^*(\Sigma_g, \mathcal{O}(-np))} M_{n,\alpha} \quad (\text{B.22})$$

Thus the absolute value of the ratio of determinants is

$$\begin{aligned} & \left| \frac{\text{Det}(i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}}{\text{Det}(i\mathcal{L}_\phi)_{\Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}} \frac{\text{Det}(-i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}}{\text{Det}(-i\mathcal{L}_\phi)_{\Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}} \right| \\ &= \exp \sum_{n,\alpha} (\chi(\mathcal{O}(-np) \otimes V_\mathfrak{k}) - \chi(K \otimes \mathcal{O}(-np) \otimes V_\mathfrak{k})) \log M_{n,\alpha} \\ &= \exp \sum_{n,\alpha} \chi(\Sigma_g) \log M_{n,\alpha} = \prod_\alpha \prod_n (2\pi n + i\alpha(\phi))^{\chi(\Sigma_g)}. \end{aligned} \quad (\text{B.23})$$

Here we recognise the infinite product representation of  $\sin^2 i\alpha(\phi)/2$ . Comparing with (4.8),

$$T_{S^1}(\phi) = \det_{\mathfrak{k}}(1 - \text{Ad } e^\phi) = \prod_{\alpha > 0} (1 - e^{\alpha(\phi)})(1 - e^{-\alpha(\phi)}) \quad (\text{B.24})$$

we deduce that

$$\left| \frac{\text{Det}(i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}}{\text{Det}(i\mathcal{L}_\phi)_{\Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}} \frac{\text{Det}(-i\mathcal{L}_\phi)_{\Omega^0(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}}{\text{Det}(-i\mathcal{L}_\phi)_{\Omega^{(1,0)}(\Sigma_g, \mathcal{O}(-np) \otimes V_\mathfrak{k})}} \right| = \mathcal{N} T_{S^1}(\phi)^{\chi(\Sigma_g)} \quad (\text{B.25})$$

where  $\mathcal{N}$  is one of the constants that we do not need to keep track of since we are independently normalising the path integral.

**The phase.** Now we come to the calculation of the phase of the determinant, in particular the shift in the level  $k$ . Formally, the calculation is similar to an analogous calculation in [4]. However, due to our gauge fixing, as opposed to non-Abelian localisation, and because we also need to find the correction to the  $\phi^2$ -term in the action, some of the details of the calculation are quite different.

The phase of the products of the ratios of determinants in (B.16) is the sum of the phases, i.e.

$$\begin{aligned} \eta_\phi(s) &= \eta_{(0,1)}(i\mathcal{L}_\phi)(s) + \eta_{(1,0)}(-i\mathcal{L}_\phi)(s) \\ &= -\frac{1}{2} \sum_{n,\alpha} (\chi(\mathcal{O}(-np) \otimes V_\alpha) + \chi(K \otimes \mathcal{O}(-np) \otimes V_\alpha)) \frac{\text{sign}(2\pi n + i\alpha(\phi))}{|2\pi n + i\alpha(\phi)|^s} \\ &= -\sum_{n,\alpha} (c_1(\mathcal{O}(-np)) + c_1(V_\alpha)) \frac{\text{sign}(2\pi n + i\alpha(\phi))}{|2\pi n + i\alpha(\phi)|^s}. \end{aligned} \quad (\text{B.26})$$

Without loss of generality we choose  $\phi$  such that  $0 < i\alpha(\phi) < 2\pi$  for the positive roots, so that

$$\begin{aligned} \eta_\phi(s) = & -2 \sum_{\alpha>0} c_1(V_\alpha) |i\alpha(\phi)|^{-s} - 2 \sum_{n \geq 1} \sum_{\alpha>0} (c_1(\mathcal{O}(-np)) + c_1(V_\alpha)) (2\pi n + i\alpha(\phi))^{-s} \\ & - 2 \sum_{n \geq 1} \sum_{\alpha>0} (c_1(\mathcal{O}(-np)) - c_1(V_\alpha)) (2\pi n - i\alpha(\phi))^{-s}. \end{aligned} \quad (\text{B.27})$$

The Riemann  $\zeta$ -function,  $\zeta(s) = \sum_{n \geq 1} n^{-s}$ , is regular on the real axis away from  $s = 1$ , satisfies  $\zeta(-1) = -1/12$ ,  $\zeta(0) = -1/2$  and behaves as

$$\zeta(s+1) = \frac{1}{s} + \gamma_0 + s\gamma_1 + \dots \quad (\text{B.28})$$

as  $s$  approaches 0. This means that if we expand  $(2\pi n \pm i\alpha(\phi))^{-s}$  in a Taylor series in  $\alpha$  we need only keep terms up to quadratic order in  $\alpha(\phi)$ . We need to keep quadratic terms since  $c_1(\mathcal{O}(-np)) = -np$ . Consequently we have

$$\begin{aligned} \eta_\phi(s) = & -2 \sum_{\alpha>0} c_1(V_\alpha) - \frac{p}{6} (\dim G - \dim T) \\ & + \frac{2}{\pi} \sum_{\alpha>0} c_1(V_\alpha) i\alpha(\phi) + \frac{p}{2\pi^2} \sum_{\alpha>0} i\alpha(\phi) i\alpha(\phi) + \mathcal{O}(s) \end{aligned} \quad (\text{B.29})$$

As  $s \rightarrow 0$  all other terms vanish.

The other ratio of determinants (4.4) has a phase which depends neither on  $\phi$  nor on the connection and the constant mode is not included. So following through the same steps, there is now no sum over  $\alpha$ , and we see that the corresponding  $\eta$ -invariant is

$$\begin{aligned} \eta_0(s) = & -\dim T \sum_{n \neq 0} c_1(\mathcal{O}(-np)) \frac{\text{sign } 2\pi n}{|2\pi n|^s} = 2p \dim T \sum_{n>0} n^{1-s} \\ \xrightarrow{s \rightarrow 0} & -\frac{1}{6} p \dim T. \end{aligned} \quad (\text{B.30})$$

Putting the pieces together, making use of (B.20) and rewriting the terms involving the fields  $\phi$  and  $F_A$  as integrals over  $\Sigma_g$ , we find that the phase of the ratio of the (square roots of) determinants (4.2, 4.4) is

$$-\frac{i\pi}{2} (\eta_\phi(0) + \eta_0(0)) = \frac{ic_{\mathfrak{g}}}{4\pi} \int_{\Sigma_g} (p \text{Tr } \phi^2 + 2 \text{Tr } \phi F_A) + 4\pi i p \Phi_0 - \int_{\Sigma_g} \rho(F_H) \quad (\text{B.31})$$

The first term corresponds to a shift of the level of the action,

$$\frac{k}{4\pi} \int_{\Sigma_g} (p \text{Tr } \phi^2 \omega + 2 \text{Tr } \phi F_A) \rightarrow \frac{k + c_{\mathfrak{g}}}{4\pi} \int_{\Sigma_g} (p \text{Tr } \phi^2 \omega + 2 \text{Tr } \phi F_A) \quad (\text{B.32})$$

The second term is an overall  $p$ -dependent phase, with

$$\Phi_0 = \frac{1}{48} \dim G, \quad (\text{B.33})$$

whose significance (framing) we discuss in section 8. Finally, the third term, which arises from

$$-i\pi \sum_{\alpha>0} c_1(V_\alpha) = - \int_{\Sigma_g} \frac{1}{2} \sum_{\alpha>0} \alpha(F_H) \equiv - \int_{\Sigma_g} \rho(F_H), \tag{B.34}$$

makes no contribution for simply-connected groups for which  $\rho(F_H)$  is an integral multiple of  $2\pi i$ . However, for non-simply connected groups this term is potentially non-trivial and cannot be neglected (see e.g. [21]).

### C. The expectation value of the Hopf link

We arrived at the expectation value for the Hopf link results by using (7.3) one can reasonably ask what we would find if we had used (7.1) instead. We consider here the case where  $M = S^3$  and  $G = SU(2)$  so that characters of the representations are, (with  $\phi$  now the coefficient in a basis of simple roots)

$$\chi_j(\phi) = \frac{\sin(j+1)\phi}{\sin \phi}. \tag{C.1}$$

A simple calculation, by writing each sine as a difference of phases, allows us to determine  $K$ ,

$$\begin{aligned} K_{ij} &= \langle \chi_i(\phi) \chi_j(\phi) \rangle \\ &= \frac{2}{\pi} e^{4\pi i p \Phi_0} \int_{-\infty}^{\infty} d\phi \sin((i+1)\phi) \sin((j+1)\phi) \exp\left(-i \frac{k+2}{2\pi} \phi^2\right) \\ &= e^{2\pi i(\Phi(i) + \Phi(j))} S_{ij} \\ &= T(i) S_{ij} T(j) \end{aligned} \tag{C.2}$$

where

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin \frac{(i+1)(j+1)\pi}{k+2} \tag{C.3}$$

and the phase is

$$\Phi(j) = \frac{c_2(j)}{(k+2)} - \frac{3}{16} + p\Phi_0 + \frac{1}{4(k+2)} = \frac{c_2(j)}{(k+2)} - \frac{k}{8(k+2)} \tag{C.4}$$

$c_2(j) = j(j+2)/4$  and  $T(j)$  agrees with the formula that was given in (7.11). We have thus obtained (8.6) without using (7.14).

### References

- [1] M. Blau and G. Thompson, *Derivation of the Verlinde formula from Chern-Simons theory and the G/G model*, *Nucl. Phys.* **B 408** (1993) 345 [[hep-th/9305010](#)].
- [2] M. Blau and G. Thompson, *Lectures on 2d gauge theories: topological aspects and path integral techniques*, in *emphProceedings of the 1993 Trieste Summer School on High Energy Physics and Cosmology*, E. Gava et al. eds., World Scientific, Singapore, 1994, p. 175-244, [hep-th/9310144](#).

- [3] M. Blau and G. Thompson, *Localization and diagonalization: a review of functional integral techniques for low dimensional gauge theories and topological field theories*, *J. Math. Phys.* **36** (1995) 2192 [[hep-th/9501075](#)].
- [4] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, [hep-th/0503126](#).
- [5] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [[hep-th/9204083](#)].
- [6] C. Vafa, *Two dimensional Yang-Mills, black holes and topological strings*, [hep-th/0406058](#).
- [7] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, *Black holes, Q-deformed 2D Yang-Mills and non-perturbative topological strings*, *Nucl. Phys.* **B 715** (2005) 304 [[hep-th/0411280](#)].
- [8] R. Lawrence and L. Rozansky, *Witten-Reshetikhin-Turaev invariants of Seifert manifolds*, *Commun. Math. Phys.* **205** (1999) 287.
- [9] M. Marino, *Chern-Simons theory, matrix integrals and perturbative three-manifold invariants*, *Commun. Math. Phys.* **253** (2004) 25 [[hep-th/0207096](#)].
- [10] E. Buffenoi and P. Roche, *Two-dimensional lattice gauge theory based on a quantum group*, *Commun. Math. Phys.* **170** (1995) 669 [[hep-th/9405126](#)].
- [11] C. Klimčík, *The formulae of Kontsevich and Verlinde from the perspective of the Drinfeld double*, *Commun. Math. Phys.* **217** (2001) 203 [[hep-th/9911239](#)].
- [12] S. de Haro, *Chern-Simons theory in lens spaces from 2D Yang-Mills on the cylinder*, *JHEP* **08** (2004) 041 [[hep-th/0407139](#)].
- [13] X. Arsiwalla, R. Boels, M. Marino and A. Sinkovics, *Phase transitions in Q-deformed 2d Yang-Mills theory and topological strings*, *Phys. Rev.* **D 73** (2006) 026005 [[hep-th/0509002](#)];  
D. Jafferis and J. Marsano, *A DK phase transition in q-deformed Yang-Mills on  $S^2$  and topological strings*, [hep-th/0509004](#);  
N. Caporaso et al., *Topological strings and large-N phase transitions, I. Nonchiral expansion of MBF Q-deformed Yang-Mills theory*, *JHEP* **01** (2006) 035 [[hep-th/0509041](#)]; *Topological strings and large-N phase transitions, II. Chiral expansion of Q-deformed Yang-Mills theory*, *JHEP* **01** (2006) 036 [[hep-th/0511043](#)].
- [14] S. de Haro, *A note on knot invariants and Q-deformed 2D Yang-Mills*, *Phys. Lett.* **B 634** (2006) 78 [[hep-th/0509167](#)].
- [15] M. Blau and G. Thompson, *On diagonalization in  $MAP(M,G)$* , *Commun. Math. Phys.* **171** (1995) 639 [[hep-th/9402097](#)].
- [16] J. Martinet, *Formes de Contact sur les varietés de dimension 3*, Springer Lecture Notes in Mathematics 209, 1971, pp. 142.
- [17] R. Bott and L. Tu, *Differential forms in algebraic topology*, Springer, 1982.
- [18] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften 298, Springer, 1991.
- [19] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.
- [20] M. Aganagic, A. Klemm, M. Marino and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, *JHEP* **02** (2004) 010 [[hep-th/0211098](#)].

- [21] M. Blau, F. Hussain and G. Thompson, *Some general aspects of coset models and topological Kazama-Suzuki models*, *Nucl. Phys. B* **488** (1997) 541 [[hep-th/9510187](#)]; *Grassmannian topological Kazama-Suzuki models and cohomology*, *Nucl. Phys. B* **488** (1997) 599 [[hep-th/9510194](#)].
- [22] E. Witten, *Topology changing amplitudes in (2+1)-dimensional gravity*, *Nucl. Phys. B* **323** (1989) 113.
- [23] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, *Phys. Rept.* **209** (1991) 129.